SOME THEOREMS ON
THE ROGERS–RAMANUJAN CONTINUED FRACTION
IN RAMANUJAN'S LOST NOTEBOOK

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ABSTRACT. In his first two letters to G. H. Hardy and in his notebooks, Ramanujan recorded many theorems about the Rogers–Ramanujan continued fraction. In his lost notebook, he offered several further assertions. The purpose of this paper is to provide proofs for many of the claims about the Rogers–Ramanujan and generalized Rogers–Ramanujan continued fractions found in the lost notebook. These theorems involve, among other things, modular equations, transformations, zeros, and class invariants.

1. INTRODUCTION

The Rogers–Ramanujan continued fraction, defined by

$$(1.1) \qquad R(q) := \frac{q^{1/5}}{1} \underset{+}{} \frac{q}{1} \underset{+}{} \frac{q^2}{1} \underset{+}{} \frac{q^3}{1} \underset{+}{} \cdots, \qquad |q| < 1,$$

first appeared in a paper by L. J. Rogers [27] in 1894. Using the Rogers–Ramanujan identities, established for the first time in [27], Rogers proved that

$$(1.2) \qquad R(q) = q^{1/5} \frac{(q; q^5)_\infty (q^4; q^5)_\infty}{(q^2; q^5)_\infty (q^3; q^5)_\infty},$$

where we employ the customary notation

$$(a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k), \qquad |q| < 1.$$

In the sequel, we shall also use the notation

$$(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k).$$

At times, we shall put $(a)_n := (a; q)_n$.

In his first two letters to G. H. Hardy [25, pp. xxvii, xxviii], [11, pp. 29, 57], Ramanujan communicated several theorems about $R(q)$. He also briefly mentioned the more general continued fraction

$$(1.3) \qquad R(a, q) := \frac{1}{1} \underset{+}{} \frac{aq}{1} \underset{+}{} \frac{aq^2}{1} \underset{+}{} \frac{aq^3}{1} \underset{+}{} \cdots, \qquad |q| < 1,$$

which we call the generalized Rogers–Ramanujan continued fraction. In his notebooks [24], Ramanujan offered many beautiful theorems about $R(q)$. In particular,

see (1.6) and (1.7) below, K. G. Ramanathan's papers [20]–[23], the memoir by
G. E. Andrews, B. C. Berndt, L. Jacobsen, and R. L. Lamphere [4], and Berndt's
book [6, Chap. 32].

Ramanujan's lost notebook [26] contains many further alluring and remarkable
results on the Rogers–Ramanujan continued fraction, and some of these have been
proved by Andrews [1], [2], Berndt and H. H. Chan [7], Berndt, Chan, and L.–
C. Zhang [8], Huang [12], S.–Y. Kang [13], [14], S. Raghavan [18], Raghavan and
S. S. Rangachari [19], Ramanathan [20]–[23], and Son [29], [30]. The purpose
of this paper is to prove several additional claims made by Ramanujan in his lost
notebook [26] about the Rogers–Ramanujan continued fraction. There exist further
generalizations of $R(q)$ and $R(a, q)$ found in the lost notebook. In particular, see
Andrews' paper [1]. However, in this paper, we primarily confine our attention to
$R(q)$, $R(a, q)$, and finite versions of both $R(q)$ and $R(a, q)$.

We now briefly describe some of the results proved herein.

In his first letter to Hardy [25, p. xxvii], [11, p. 29], Ramanujan claimed that
$R^5(q)$ is a particular quotient of quartic polynomials in $R(q^5)$. This was first proved
in print by Rogers [28] in 1920, while G. N. Watson [32] gave another proof nine
years later. At scattered places in his notebooks [24], Ramanujan also gave modular
equations relating $R(q)$ with $R(-q)$, $R(q^2)$, $R(q^3)$, and $R(q^4)$. In the publication of
his lost notebook [26], these results are conveniently summarized by Ramanujan on
page 365. Proofs of most of these modular relations can be found in the memoir [4]
(Entries 6, 20, 21, and 24–26 on pages 11, 27, 28, and 31–37) and in Berndt's book
[6, Chap. 32, Entries 1–6]. Rogers [28] found modular equations relating $R(q)$ with
$R(q^n)$, for $n = 2, 3, 5$, and 11; the latter equation is not found in Ramanujan's work.
On page 205 in his lost notebook [26], Ramanujan offers two modular equations
relating the Rogers–Ramanujan continued fraction at *three* arguments. These, and
a few other modular equations of the same sort are proved in Section 2.

To describe the next two theorems, we need to define Ramanujan's general theta–
function $f(a, b)$, namely,

$$(1.4) \qquad f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1.$$

In particular, set

$$(1.5) \qquad f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty, \qquad |q| < 1.$$

The latter equality is Euler's pentagonal number theorem.

Two of the most important formulas for $R(q)$ are given by

$$(1.6) \qquad \frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)}$$

and

$$(1.7) \qquad \frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{q f^6(-q^5)}.$$

These equalities were found by Watson in Ramanujan's notebooks and proved by
him [32] in order to establish claims [32], [33] about the Rogers–Ramanujan con-
tinued fraction communicated by Ramanujan in the aforementioned two letters to
Hardy. Different proofs of (1.6) and (1.7) can be found in Berndt's book [5, pp.

265–267] and Son's paper [30]. Berndt, Chan, and Zhang [8] recently employed (1.6) and (1.7) in developing formulas for the explicit evaluation of $R(q)$, in particular, for the values of $R(q)$ claimed by Ramanujan in his lost notebook.

On page 48 in his lost notebook, Ramanujan offers two further formulas akin to (1.6) and (1.7). These formulas are "between" (1.6) and (1.7) in that they involve $R^2(q)$ and $R^3(q)$. Statements and proofs of these identities can be found in Section 3.

On the other hand, on page 206 in his lost notebook, Ramanujan claims that (1.6) and (1.7) can be refined by factoring each side into two factors and then equating appropriate factors on each side, giving four equalities. It is amazing that factoring in this way actually leads to identities. After proving these identities, given in Section 4, we discovered that Ramanathan [20] had published a proof. However, possibly due to an attempt to be brief, the argument for a key step is absent. This important step, an application of an addition theorem for theta–functions due to Ramanujan and found in Ramanujan's notebooks [24], is perhaps the most difficult part of the proof, and so it seems worthwhile to give a complete proof here.

Most of the results in Sections 2–4 can also be proved by using the theory of modular forms. However, we prefer to give more elementary proofs, more in the spirit of Ramanujan and, in our opinion, more instructive as well.

In Section 5, we utilize the Bauer–Muir transformation to prove the following remarkable transformation for a generalization of the Rogers–Ramanujan continued fraction. Let $\alpha = (1 + \sqrt{1 + 4k})/2$ and $\beta = (-1 + \sqrt{1 + 4k})/2$, where $k \geq 0$. Then

$$\frac{1}{1+} \frac{k+q}{1} + \frac{k+q^2}{1} + \frac{k+q^3}{1} + \cdots = \frac{1}{\alpha+} \frac{q}{\alpha + \beta q+} \frac{q^2}{\alpha + \beta q^2+} \frac{q^3}{\alpha + \beta q^3+} + \cdots .$$

On page 48 in his lost notebook, Ramanujan examines the zeros of $1/R(-a, q)$, where $R(a, q)$ is defined in (1.3). In particular, if $a > 0$, Ramanujan derives approximations and an asymptotic expansion in terms of descending powers of a for the smallest real zero q_0. See Section 6 for a description of this work.

In Section 7, we state without proofs some formulas of Ramanujan arising from two of his modular equations.

Section 8 offers an identity for a finite generalized Rogers–Ramanujan continued fraction.

Finally, in Section 9, we examine certain finite Rogers–Ramanujan continued fractions. Ramanujan asserted that certain zeros of these continued fractions can be expressed in terms of class invariants or singular moduli. Such phenomena appear to be rare, and apparently no general theorems exist.

Page numbers in theorem headings refer to the lost notebook.

2. Modular Equations

Recall that $R(q)$ is defined in (1.1). Following Ramanujan, set

$$u = R(q), \qquad u' = -R(-q), \qquad v = R(q^2), \qquad w = R(q^4).$$

Theorem 2.1 (p. 205). *We have*

(2.1)
$$uw = \frac{w - u^2 v}{w + v^2}$$

and

$$(2.2) \qquad uu'v^2 = \frac{uu' - v}{u' - u}.$$

Proof. First recall that

$$(2.3) \qquad uv^2 = \frac{v - u^2}{v + u^2}.$$

This modular equation is found in Ramanujan's notebooks [24, vol. 2, p. 326] and with the publication of his lost notebook [26, p. 365, Entry (10)(a)]. The first proof was given in [4, p. 31, Entry 24(i)] and can also be found in [6, Chap. 32, Entry 1]. Replacing q by q^2 in (2.3), we find that

$$(2.4) \qquad vw^2 = \frac{w - v^2}{w + v^2}.$$

Rewriting (2.3) and (2.4) in the forms

$$(2.5) \qquad uv^3 + u^3v^2 - v + u^2 = 0,$$

$$(2.6) \qquad w^2v^3 + v^2 + w^3v - w = 0,$$

respectively, we eliminate the constant terms in this pair of cubic equations in v by multiplying (2.5) by w and (2.6) by u^2 and then adding the resulting equalities. Accordingly,

$$v\left((uw + u^2w^2)v^2 + (u^3w + u^2)v + w(u^2w^2 - 1)\right)$$
$$= v(1 + uw)\left(uwv^2 + u^2v + w(uw - 1)\right) = 0.$$

Since, for $0 < q < 1$, $v(1 + uw) \neq 0$, we conclude that

$$(2.7) \qquad uwv^2 + u^2v + w(uw - 1) = 0.$$

A rearrangement of (2.7) yields (2.1).

Second, replace q by $-q$ in (2.3) to deduce that

$$(2.8) \qquad -u'v^2 = \frac{v - u'^2}{v + u'^2}.$$

Rewriting (2.3) and (2.8) in the forms

$$(2.9) \qquad v - u^2 = uv^2(v + u^2),$$

$$(2.10) \qquad v - u'^2 = -u'v^2(v + u'^2),$$

respectively, we multiply (2.9) by u', multiply (2.10) by u, and add the resulting equations to eliminate the cubic term in v. Thus,

$$v(u + u') - uu'(u + u') = uu'v^2(u^2 - u'^2) = uu'v^2(u + u')(u - u').$$

Since $u + u' \neq 0$, for $0 < q < 1$,

$$(2.11) \qquad v - uu' = uu'v^2(u - u').$$

We now immediately deduce (2.2) from (2.11). $\qquad\qquad\qquad\qquad$ □

The modular equations in Theorems 2.2–2.4 are in the spirit of (2.1) and (2.2) but are not found in Ramanujan's work.

Theorem 2.2. *We have*

$$(2.12) \qquad u'w = \frac{u'^2 v - w}{v^2 + w}$$

and

$$(2.13) \qquad -vw = \frac{u'(v^2 - w)}{u'^2 v - w}.$$

We only sketch the proofs of (2.12) and (2.13), as they are similar to those for (2.1) and (2.2). We use (2.4) and (2.8) in both proofs. To obtain (2.12), we eliminate the constant terms in the two cubic polynomials in v, while to establish (2.13), we eliminate the cubic terms from the same pair of equations.

Theorem 2.3. *We have*

$$uu'v = \frac{u' - u}{v + uu'}.$$

To prove Theorem 2.3, employ (2.9) and (2.10) and proceed as in the previous proofs.

Theorem 2.4. *We have*

$$vw = \frac{u(v^2 - w)}{u^2 v - w}.$$

To prove Theorem 2.4, utilize (2.5) and (2.6).

3. Two Identities for $R(q)$

Theorem 3.1 (p. 48). *If $f(-q)$ is defined by (1.5), then*

$$(3.1) \qquad \sum_{n=-\infty}^{\infty} (-1)^n (10n + 3) q^{(5n+3)n/2} = \left(\frac{3}{R^2(q)} + R^3(q) \right) q^{2/5} f^3(-q^5)$$

and

$$(3.2) \qquad \sum_{n=-\infty}^{\infty} (-1)^n (10n + 1) q^{(5n+1)n/2} = \left(\frac{1}{R^3(q)} - 3R^2(q) \right) q^{3/5} f^3(-q^5).$$

Proof. The key to our proofs is Jacobi's identity [5, p. 39, Entry 24(ii)],

$$(3.3) \qquad f^3(-q) = \sum_{n=-\infty}^{\infty} (-1)^n n q^{n(n+1)/2}.$$

By (1.6),

$$\left(\frac{1}{R(q)} - 1 - R(q) \right)^3 = \frac{f^3(-q^{1/5})}{q^{3/5} f^3(-q^5)},$$

from which it follows that

$$(3.4) \quad q^{3/5} f^3(-q^5) \left\{ 5 - \left(\frac{3}{R^2(q)} + R^3(q) \right) + \left(\frac{1}{R^3(q)} - 3R^2(q) \right) \right\} = f^3(-q^{1/5}).$$

If we expand the left side of (3.4) as a power series in q, we find that the exponents of q in

$$(3.5) \qquad 5q^{3/5} f^3(-q^5)$$

are $\equiv \frac{3}{5}$ (mod 1), the exponents in

$$(3.6) \qquad -q^{3/5} f^3(-q^5) \left(\frac{3}{R^2(q)} + R^3(q) \right)$$

are $\equiv \frac{1}{5}$ (mod 1), and the exponents in

$$(3.7) \qquad q^{3/5} f^3(-q^5) \left(\frac{1}{R^3(q)} - 3R^2(q) \right)$$

are integers.

By Jacobi's identity (3.3),

$$
\begin{aligned}
f^3(-q^{1/5}) &= \sum_{n=-\infty}^{\infty} (-1)^n n q^{n(n+1)/10} \\
&= \sum_{n=-\infty}^{\infty} (-1)^{5n}(5n)q^{5n(5n+1)/10} \\
&\quad + \sum_{n=-\infty}^{\infty} (-1)^{5n+1}(5n+1)q^{(5n+1)(5n+2)/10} \\
(3.8) &\quad + \sum_{n=-\infty}^{\infty} (-1)^{5n+2}(5n+2)q^{(5n+2)(5n+3)/10} \\
&\quad + \sum_{k=-\infty}^{\infty} (-1)^{5k+3}(5k+3)q^{(5k+3)(5k+4)/10} \\
&\quad + \sum_{k=-\infty}^{\infty} (-1)^{5k+4}(5k+4)q^{(5k+4)(5k+5)/10}.
\end{aligned}
$$

Letting $k = -n - 1$, we obtain

(3.9)

$$\sum_{k=-\infty}^{\infty} (-1)^{5k+3}(5k+3)q^{(5k+3)(5k+4)/10} = -\sum_{n=-\infty}^{\infty} (-1)^n(5n+2)q^{(5n+2)(5n+1)/10}$$

and

(3.10)

$$\sum_{k=-\infty}^{\infty} (-1)^{5k+4}(5k+4)q^{(5k+4)(5k+5)/10} = \sum_{n=-\infty}^{\infty} (-1)^n(5n+1)q^{(5n+1)(5n)/10}.$$

Therefore, substituting (3.9) and (3.10) in (3.8), we find that

(3.11)

$$
\begin{aligned}
f^3(-q^{1/5}) &= \sum_{n=-\infty}^{\infty} (-1)^n \Big(5n + (5n+1) \Big) q^{n(5n+1)/2} \\
&\quad - q^{1/5} \sum_{n=-\infty}^{\infty} (-1)^n \Big((5n+1) + (5n+2) \Big) q^{(5n+3)n/2} \\
&\quad + q^{3/5} \left(5 \sum_{n=-\infty}^{\infty} (-1)^n n (q^5)^{(n+1)n/2} + 2 \sum_{n=-\infty}^{\infty} (-1)^n (q^5)^{(n+1)n/2} \right).
\end{aligned}
$$

As

$$\sum_{n=-\infty}^{\infty} (-1)^n (q^5)^{(n+1)n/2} = 0$$

and, by (3.3),

$$\sum_{n=-\infty}^{\infty} (-1)^n n (q^5)^{n(n+1)/2} = f^3(-q^5),$$

we find that, by (3.11),

(3.12)
$$f^3(-q^{1/5}) = \sum_{n=-\infty}^{\infty} (-1)^n (10n+1) q^{n(5n+1)/2}$$
$$- q^{1/5} \sum_{n=-\infty}^{\infty} (-1)^n (10n+3) q^{(5n+3)n/2}$$
$$+ 5q^{3/5} f^3(-q^5).$$

The powers of q in the first sum on the right side of (3.12) are integers, the powers of q in the second expression are $\equiv \frac{1}{5} \pmod 1$, while the powers of q in the last expression on the right side of (3.12) are $\equiv \frac{3}{5} \pmod 1$. Therefore, from our observations about the powers of q in (3.5)–(3.7) and our observations about the powers of q in (3.12), we conclude that

$$-q^{3/5} f^3(-q^5) \left(\frac{3}{R^2(q)} + R^3(q) \right) = -q^{1/5} \sum_{n=-\infty}^{\infty} (-1)^n (10n+3) q^{(5n+3)n/2}$$

and

$$q^{3/5} f^3(-q^5) \left(\frac{1}{R^3(q)} - 3R^2(q) \right) = \sum_{n=-\infty}^{\infty} (-1)^n (10n+1) q^{n(5n+1)/2}.$$

The identities (3.1) and (3.2) now follow, respectively, from the last two equalities. $\qquad \square$

4. Refinements of Ramanujan's Identities (1.6) and (1.7)

Theorem 4.1 (p. 206). *Let $t = R(q)$, and set $\alpha = \frac{1-\sqrt{5}}{2}$ and $\beta = \frac{1+\sqrt{5}}{2}$. Then*

(4.1)
$$\frac{1}{\sqrt{t}} - \alpha\sqrt{t} = \frac{1}{q^{1/10}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{n=1}^{\infty} \frac{1}{1 + \alpha q^{n/5} + q^{2n/5}},$$

(4.2)
$$\frac{1}{\sqrt{t}} - \beta\sqrt{t} = \frac{1}{q^{1/10}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{n=1}^{\infty} \frac{1}{1 + \beta q^{n/5} + q^{2n/5}},$$

(4.3)
$$\left(\frac{1}{\sqrt{t}} \right)^5 - \left(\alpha\sqrt{t} \right)^5 = \frac{1}{q^{1/2}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{n=1}^{\infty} \frac{1}{(1 + \alpha q^n + q^{2n})^5},$$

(4.4)
$$\left(\frac{1}{\sqrt{t}} \right)^5 - \left(\beta\sqrt{t} \right)^5 = \frac{1}{q^{1/2}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{n=1}^{\infty} \frac{1}{(1 + \beta q^n + q^{2n})^5}.$$

It is not difficult to verify that by multiplying (4.1) by (4.2) we obtain (1.6), and by multiplying (4.3) by (4.4) we obtain (1.7). Therefore, (4.1) and (4.3) are equivalent to (4.2) and (4.4), respectively, and so it suffices to establish (4.1) and (4.3).

Lemma 4.2. *We have*

$$f(-1, a) = 0, \tag{4.5}$$

and, if n is an integer,

$$f(a, b) = a^{n(n+1)/2} b^{n(n-1)/2} f\left(a(ab)^n, b(ab)^{-n}\right). \tag{4.6}$$

For proofs of these elementary properties, see [5, p. 34, Entry 18].

Lemma 4.3 (Jacobi's Triple Product Identity). *If $f(a, b)$ is defined by (1.4), then*

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty.$$

For a proof, see [5, p. 35, Entry 19].

Corollary 4.4.

$$f(-q, -q^4) f(-q^2, -q^3) = f(-q) f(-q^5).$$

This follows immediately from Lemma 4.3 and (1.5). See also [5, p. 44, Corollary].

Lemma 4.5. *Let $U_n = a^{n(n+1)/2} b^{n(n-1)/2}$ and $V_n = a^{n(n-1)/2} b^{n(n+1)/2}$. Then*

$$f(U_1, V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right).$$

For a proof of Lemma 4.5, see [5, p. 48, Entry 31].

Corollary 4.6. *If $\zeta = e^{2\pi i/5}$, then*

$$f(-q^2, -q^3) - \alpha q^{1/5} f(-q, -q^4) = f(-\zeta^2, -\zeta^3 q^{1/5})/(1 - \zeta^2) \tag{4.7}$$

and

$$f(-q^2, -q^3) - \beta q^{1/5} f(-q, -q^4) = f(-\zeta, -\zeta^4 q^{1/5})/(1 - \zeta). \tag{4.8}$$

Proof. By Lemma 4.5 with $n = 5$, $a = -\zeta^2$, and $b = -\zeta^3 q^{1/5}$,

$$f(-\zeta^2, -\zeta^3 q^{1/5}) = f(-q^2, -q^3) - \zeta^2 f(-q^3, -q^2) + \zeta^4 q^{1/5} f(-q^4, -q)$$
$$- \zeta q^{3/5} f(-q^5, -1) + \zeta^3 q^{6/5} f(-q^6, -q^{-1})$$
$$= (1 - \zeta^2) f(-q^2, -q^3) - (\zeta^3 - \zeta^4) q^{1/5} f(-q, -q^4),$$

since $f(-q^5, -1) = 0$ and $f(-q^6, -q^{-1}) = -q^{-1} f(-q, -q^4)$ by Lemma 4.2, with $a = -q^{-1}$, $b = -q^6$ and $n = 1$ in (4.6). Finally, (4.7) follows easily by noting that $\alpha = -(\zeta + \zeta^{-1})$ and so $\zeta^3 - \zeta^4 = \alpha(1 - \zeta^2)$.

By Lemma 4.5 with $n = 5$, $a = -\zeta$, and $b = -\zeta^4 q^{1/5}$, and the observations made above,

$$f(-\zeta, -\zeta^4 q^{1/5}) = f(-q^2, -q^3) - \zeta f(-q^3, -q^2) + \zeta^2 q^{1/5} f(-q^4, -q)$$
$$- \zeta^3 q^{3/5} f(-q^5, -1) + \zeta^4 q^{6/5} f(-q^6, -q^{-1})$$
$$= (1 - \zeta)\left(f(-q^2, -q^3) - \beta q^{1/5} f(-q, -q^4)\right),$$

since $\zeta^2 + \zeta^3 = -\beta$. This proves (4.8).   □

Lemma 4.7. *Let n be a positive integer not divisible by 5, and set $\zeta = e^{2\pi i/5}$.*
Then

$$\prod_{j=0}^{4}(1 + \alpha\zeta^{nj}q^{n/5} + \zeta^{2nj}q^{2n/5}) = (1 - q^n)^2.$$

Proof. First, recall that $\alpha = -(\zeta + \zeta^{-1})$. Then,

$$\prod_{j=0}^{4}(1 + \alpha\zeta^{nj}q^{n/5} + \zeta^{2nj}q^{2n/5}) = \prod_{j=0}^{4}\left(1 - (\zeta + \zeta^{-1})\zeta^{nj}q^{n/5} + \zeta^{2nj}q^{2n/5}\right)$$

$$= \left\{\prod_{j=0}^{4}(1 - \zeta^{nj-1}q^{n/5})\right\}\left\{\prod_{j=0}^{4}(1 - \zeta^{nj+1}q^{n/5})\right\}.$$

Since n is not divisible by 5, ζ^{nj} runs through all the 5-th roots of unity when j runs through 0, 1, 2, 3, 4. Therefore, the last two products are both equal to

$$\prod_{j=0}^{4}(1 - \zeta^j q^{n/5}) = 1 - q^n.$$

This completes the proof. $\qquad\square$

Proof of Theorem 4.1. Let ζ denote $e^{2\pi i/5}$. By (1.2), (4.7), and Corollary 4.4,

$$\frac{1}{\sqrt{t}} - \alpha\sqrt{t} = \frac{f(-q^2, -q^3) - \alpha q^{1/5}f(-q, -q^4)}{q^{1/10}\sqrt{f(-q, -q^4)f(-q^2, -q^3)}}$$

$$(4.9) \qquad\qquad = \frac{f(-\zeta^2, -\zeta^3 q^{1/5})/(1 - \zeta^2)}{q^{1/10}\sqrt{f(-q)f(-q^5)}}.$$

By Lemma 4.3 and (1.5),

$$f(-\zeta^2, -\zeta^3 q^{1/5})/(1 - \zeta^2) = (\zeta^2 q^{1/5}; q^{1/5})_\infty(\zeta^3 q^{1/5}; q^{1/5})_\infty(q^{1/5}; q^{1/5})_\infty$$

$$= \frac{f(-q)}{(\zeta q^{1/5}; q^{1/5})_\infty(\zeta^4 q^{1/5}; q^{1/5})_\infty}$$

$$(4.10) \qquad\qquad = \frac{f(-q)}{\prod_{n=1}^{\infty}(1 + \alpha q^{n/5} + q^{2n/5})}.$$

Substituting (4.10) in (4.9), we complete the proof of (4.1).

It remains to prove (4.3). This can be done by using (4.1). For each $j = 0, 1, 2, 3, 4$, we obtain an identity by replacing $q^{1/5}$ with $\zeta^j q^{1/5}$ in (4.1). Multiplying these five identities together, we deduce that

$$\prod_{j=0}^{4}\left\{\frac{1}{\sqrt{\zeta^j t}} - \alpha\sqrt{\zeta^j t}\right\}$$

$$= \prod_{j=0}^{4}\left\{\frac{1}{(\zeta^j q^{1/5})^{\frac{1}{2}}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{1 + \alpha(\zeta^j q^{1/5})^n + (\zeta^j q^{1/5})^{2n}}\right\},$$

which can easily be reduced to

$$(4.11) \quad \left(\frac{1}{\sqrt{t}}\right)^5 - \left(\alpha\sqrt{t}\right)^5 = \frac{1}{q^{1/2}}\sqrt{\frac{f^5(-q)}{f^5(-q^5)}}\prod_{j=0}^{4}\prod_{n=1}^{\infty}\frac{1}{1 + \alpha(\zeta^j q^{1/5})^n + (\zeta^j q^{1/5})^{2n}}.$$

Furthermore, the double product in (4.11) equals

$$
\left\{ \prod_{j=0}^{4} \prod_{5|n} \frac{1}{1 + \alpha(\zeta^j q^{1/5})^n + (\zeta^j q^{1/5})^{2n}} \right\} \left\{ \prod_{j=0}^{4} \prod_{5\nmid n} \frac{1}{1 + \alpha(\zeta^j q^{1/5})^n + (\zeta^j q^{1/5})^{2n}} \right\}
$$

$$
= \left\{ \prod_{k=1}^{\infty} \frac{1}{(1 + \alpha q^k + q^{2k})^5} \right\} \left\{ \prod_{5\nmid n} \prod_{j=0}^{4} \frac{1}{1 + \alpha(\zeta^j q^{1/5})^n + (\zeta^j q^{1/5})^{2n}} \right\}
$$

$$
= \left\{ \prod_{k=1}^{\infty} \frac{1}{(1 + \alpha q^k + q^{2k})^5} \right\} \left\{ \prod_{5\nmid n} \frac{1}{(1 - q^n)^2} \right\}
$$

$$
= \left\{ \prod_{k=1}^{\infty} \frac{1}{(1 + \alpha q^k + q^{2k})^5} \right\} \frac{f^2(-q^5)}{f^2(-q)},
$$

where the penultimate equality follows from Lemma 4.7. Therefore, (4.11) becomes

$$
\left(\frac{1}{\sqrt{t}} \right)^5 - \left(\alpha\sqrt{t} \right)^5 = \frac{1}{q^{1/2}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{k=1}^{\infty} \frac{1}{(1 + \alpha q^k + q^{2k})^5}.
$$

This completes the proof of Theorem 4.1. $\square$

Alternatively, Theorem 4.1 can be proved without the help of (1.6) and (1.7). Indeed, by using (4.8) instead of (4.7), one can prove (4.2) and then (4.4) in a similar manner. By doing so, we discover a new proof for the two remarkable identities (1.6) and (1.7).

5. A Transformation Formula

Theorem 5.1 (p. 46). *Let* $k \geq 0, \alpha = (1+\sqrt{1+4k})/2$ *and* $\beta = (-1+\sqrt{1+4k})/2$. *Then, for* $|q| < 1$ *and* $\mathrm{Re}\, q > 0$,

$$
(5.1) \quad \frac{1}{1+} \frac{k+q}{1} + \frac{k+q^2}{1} + \frac{k+q^3}{1} + \cdots = \frac{1}{\alpha+} \frac{q}{\alpha + \beta q+} \frac{q^2}{\alpha + \beta q^2+} \frac{q^3}{\alpha + \beta q^3+} \cdots.
$$

This is a beautiful theorem, and we do not know how Ramanujan derived it. We shall use the Bauer–Muir transformation to establish Theorem 5.1, but it seems unlikely that Ramanujan proceeded in this way.

If $q = 0$ in (5.1), then we find that

$$
\frac{1}{1+} \frac{k}{1+} \frac{k}{1+} \frac{k}{1+} \cdots = \frac{1}{\alpha},
$$

which can be established by elementary means.

If $q = 1$ in (5.1), we find that

$$
\frac{1}{1+} \frac{k+1}{1} + \frac{k+1}{1} + \frac{k+1}{1} + \cdots = \frac{1}{\alpha+} \frac{1}{\sqrt{1+4k}+} \frac{1}{\sqrt{1+4k}+} \frac{1}{\sqrt{1+4k}+} \cdots.
$$

This identity can be easily verified by elementary computations; both sides are equal to

$$
\frac{2}{1 + \sqrt{5 + 4k}}.
$$

For $|q| > 1$, the continued fraction on the left side of (5.1) diverges. However, by Van Vleck's theorem (Lorentzen and Waadeland [15, p. 32]), the continued fraction on the right side of (5.1) converges for all q such that Re $q > 0$.

If $k = 0$, then $\alpha = 1$ and $\beta = 0$. Thus, (5.1) reduces to a tautology.

If $k = 2$, then $\alpha = 2$ and $\beta = 1$. We thus obtain the following corollary, which Ramanujan also records, but with a slight misprint.

Corollary 5.2 (p. 46). *For* $|q| < 1$,

$$\frac{1}{1+}\frac{2+q}{1}+\frac{2+q^2}{1}+\frac{2+q^3}{1}+\cdots = \frac{1}{2+}\frac{q}{2+q}+\frac{q^2}{2+q^2}+\frac{q^3}{2+q^3}+\cdots.$$

Proof of Theorem 5.1. As indicated above, we shall apply the Bauer–Muir transformation ([15, p. 76]), which we now briefly describe. Given a continued fraction $b_0 + \mathbb{K}(a_n/b_n)$ and a sequence of complex numbers $\{w_n\}, 0 \leq n < \infty$, define

$$(5.2) \qquad \lambda_n = a_n - w_{n-1}(b_n + w_n), \qquad n = 1, 2, \ldots.$$

Assume that $\lambda_n \neq 0$ for every $n \geq 1$. Let

$$(5.3) \qquad q_n = \lambda_{n+1}/\lambda_n, \qquad n \geq 1.$$

If, for $n \geq 2$,

$$(5.4) \qquad c_n = a_{n-1}q_{n-1} \qquad \text{and} \qquad d_n = b_n + w_n - w_{n-2}q_{n-1},$$

then

$$(5.5) \qquad b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots = b_0 + w_0 + \frac{\lambda_1}{b_1 + w_1} + \frac{c_2}{d_2} + \frac{c_3}{d_3} + \cdots.$$

If $C(q)$ denotes the reciprocal of the continued fraction on the left side of (5.1), and if we employ the notation on the left side of (5.5), then, for $n \geq 1$, $a_n = k+q^n$, and, for $n \geq 0$, $b_n = 1$. Now set $w_n = \beta, n \geq 0$. Then, by (5.2), since $1 + \beta = \alpha$ and $\alpha\beta = k$, it follows that $\lambda_n = q^n$. Thus, by (5.3), $q_n = q$, and, by (5.4), if $n \geq 2$, $c_n = (k + q^{n-1})q$ and $d_n = \alpha - \beta q$, since $1 + \beta = \alpha$. Also, $b_0 + w_0 = \alpha = b_1 + w_1$. Thus, by (5.5),

$$(5.6) \qquad C(q) = \alpha + \frac{q}{\alpha+} \frac{(k+q)q}{\alpha - \beta q} + \frac{(k+q^2)q}{\alpha - \beta q} + \cdots =: \alpha + \frac{q}{C_1(q)}.$$

For the continued fraction $C_1(q)$, in the notation of (5.5), $b_0 = \alpha$, $b_n = \alpha - \beta q$, and $a_n = (k + q^n)q$, for $n \geq 1$. We apply the Bauer–Muir transformation a second time. Set $w_n = \beta q, n \geq 0$. A brief calculation shows that, by (5.2), $\lambda_n = q^{n+1}$. Thus, $b_0 + w_0 = \alpha + \beta q$, $b_1 + w_1 = \alpha$, $c_n = (k + q^{n-1})q^2$, and $d_n = \alpha - \beta q^2$, where $n \geq 2$. Hence, after applying the Bauer–Muir transformation to $C_1(q)$ in (5.6), we find that

$$(5.7) \qquad \begin{aligned} C(q) &= \alpha + \frac{q}{\alpha + \beta q+} \frac{q^2}{\alpha +} \frac{(k+q)q^2}{\alpha - \beta q^2} + \frac{(k+q^2)q^2}{\alpha - \beta q^2} + \cdots \\ &=: \alpha + \frac{q}{\alpha + \beta q+} \frac{q^2}{C_2(q)}. \end{aligned}$$

Applying the Bauer–Muir transformation to $C_2(q)$ and proceeding as in the two previous applications, we find that, if $w_n = \beta q^2$, then $\lambda_n = q^{n+2}$. Thus, $b_0 + w_0 =$

$\alpha + \beta q^2$, $b_1 + w_1 = \alpha$, $c_n = (k + q^{n-1})q^3$, and $d_n = \alpha - \beta q^3$, where $n \geq 2$. Hence, from (5.7),

$$C(q) = \alpha + \cfrac{q}{\alpha + \beta q} \cfrac{q^2}{+\alpha + \beta q^2} \cfrac{q^3}{+\alpha} \cfrac{(k+q)q^3}{+\alpha - \beta q^3} \cfrac{(k+q^2)q^3}{+\alpha - \beta q^3} + \cdots$$

$$= \cdots$$

(5.8)

$$= \alpha + \cfrac{q}{\alpha + \beta q} \cfrac{q^2}{+\alpha + \beta q^2} + \cdots + \cfrac{q^{n-1}}{\alpha + \beta q^{n-1}}$$

$$+ \cfrac{q^n}{\alpha} \cfrac{(k+q)q^n}{+\alpha - \beta q^n} \cfrac{(k+q^2)q^n}{+\alpha - \beta q^n} + \cdots,$$

after an easy inductive argument on n. Letting n tend to ∞ in (5.8), we deduce (5.1). As indicated earlier, the transformed continued fraction converges for Re $q > 0$. □

Lorentzen and Waadeland [15, pp. 77–80] used the Bauer–Muir transformation to prove a special case of Theorem 5.1 and to discuss the rapidity of convergence of the transformed continued fraction. D. Bowman has informed us that he can prove Theorem 5.1 by using continued fractions for certain basis hypergeometric series and the second iterate of Heine's transformation.

6. Zeros of the Generalized Rogers–Ramanujan Continued Fraction

Theorem 6.1 (p. 48). *The smallest real zero of*

$$F(q) := 1 - \cfrac{q}{1-} \cfrac{q^2}{1-} \cfrac{q^3}{1-} \cdots$$

is approximately equal to 0.576148.

Ramanujan actually gives the value 0.5762 for this zero. He also does not indicate the possibility of other real zeros.

We considered several approaches to Ramanujan's claim, including an examination of the zeros of convergents to $F(q)$. However, only for the method described below could we obtain a proper error analysis. Ramanujan possibly used an approximating polynomial of lower degree than that below, along with an iterative procedure such as Newton's method. However, in any case, the numerical calculations seem formidable, and we wonder how Ramanujan might have proceeded.

Proof. We employ the corollary to Entry 15 in Chapter 16 in Ramanujan's second notebook [5, p. 30], namely,

(6.1)

$$\frac{\displaystyle\sum_{k=0}^{\infty} \frac{(-a)^k q^{k^2}}{(q)_k}}{\displaystyle\sum_{k=0}^{\infty} \frac{(-a)^k q^{k(k+1)}}{(q)_k}} = 1 - \cfrac{aq}{1-} \cfrac{aq^2}{1-} \cfrac{aq^3}{1-} \cdots =: F(a, q).$$

Setting $a = 1$ in (6.1), we shall examine the zeros of a partial sum of the numerator, namely,

$$\sum_{k=0}^{5} \frac{(-1)^k q^{k^2}}{(q)_k} = \frac{1}{(q)_5} \Big(1 - 2q - q^2 + q^3 + 2q^4 + 2q^5 + q^6 - q^7 - 4q^8$$

$$- 4q^9 - q^{10} + 2q^{11} + 2q^{12} + 4q^{13} + 2q^{14} - 2q^{15} - q^{18} - q^{21} - q^{25} \Big).$$

Using *Mathematica*, we find that the only real zero is approximately

$$q_0 := 0.576148762259.$$

By the alternating series test, q_0 approximates the least real zero of $F(1, q) = F(q)$ with a (positive) error less than

$$\frac{q_0^{36}}{(q_0)_6} = 1.38201727 \times 10^{-8}.$$

This completes the proof. □

The continued fraction $F(q)$ and its least positive zero $0.576148\ldots$ are important in the enumeration of "coins in a fountain" [16] and in the study of birth and death processes [17].

Theorem 6.2 (p. 48). *Let $q_0 = q_0(a)$ denote the least positive zero of $F(a, q)$, where $F(a, q)$ is defined by (6.1). Then, as a tends to ∞,*

(6.2)
$$q_0 \sim \frac{1}{a} - \frac{1}{a^2} + \frac{2}{a^3} - \frac{6}{a^4} + \frac{21}{a^5} - \frac{79}{a^6} + \frac{311}{a^7}$$
$$- \frac{1266}{a^8} + \frac{5289}{a^9} - \frac{22553}{a^{10}} + \frac{97763}{a^{11}} - \cdots .$$

Ramanujan calculated many asymptotic expansions in his notebooks, and it seems likely that in many instances, including the present one, Ramanujan employed the method of successive approximations. We also utilize this method below, but if Ramanujan also did so, he must have been able to effect the calculations more easily.

Proof. We shall calculate the first few coefficients in (6.2) by the method of successive approximations. We then describe how we used *Mathematica* for the remaining coefficients.

In view of (6.1), first set

$$1 - \frac{aq}{1} = 0.$$

Then $q = 1/a$ is a first approximation for q_0. Next, set

(6.3)
$$1 - \frac{aq}{1 -} \frac{aq^2}{1} = 0$$

and set $q = 1/a + x/a^2$ in (6.3), where x is to be determined. Then

$$1 - a\left(\frac{1}{a} + \frac{x}{a^2}\right)^2 - a\left(\frac{1}{a} + \frac{x}{a^2}\right) = 0.$$

Equating coefficients of $1/a$, we deduce that $x = -1$. Third, set

(6.4)
$$1 - \frac{aq}{1 -} \frac{aq^2}{1 -} \frac{aq^3}{1} = 0$$

and let $q = 1/a - 1/a^2 + x/a^3$ in (6.4). Equating coefficients of $1/a^2$, we deduce that $x = 2$.

Continuing in this way, we find that the calculations become increasingly more difficult. Since at each stage we are approximating the zeros of a finite continued fraction, we use an analogue of (6.1) for the finite generalized Rogers–Ramanujan

continued fraction found in Ramanujan's notebooks. Thus, for each positive integer n [5, p. 31, Entry 16],

(6.5)
$$
\frac{\displaystyle\sum_{k=0}^{[(n+1)/2]} \frac{(-a)^k q^{k^2} (q)_{n-k+1}}{(q)_k (q)_{n-2k+1}}}{\displaystyle\sum_{k=0}^{[n/2]} \frac{(-a)^k q^{k(k+1)} (q)_{n-k}}{(q)_k (q)_{n-2k}}} = 1 - \frac{aq}{1} - \frac{aq^2}{1} - \cdots - \frac{aq^n}{1}.
$$

To calculate the first eleven terms in the asymptotic expansion of q_0, we need to take $n = 11$ above. Discarding those terms which do not arise in the calculation of the first eleven coefficients, we successively approximate the zeros of

(6.6)
$$
(1-q)(1-q^2)(1-q^3)(1-q^4) - aq(1-q^{11})(1-q^2)(1-q^3)(1-q^4) \\
+ a^2 q^4 (1-q^9)(1-q^3)(1-q^4) - a^3 q^9 (1-q^4).
$$

We used *Mathematica* in (6.6) to successively calculate the coefficients of $a^{-j}, 1 \leq j \leq 11$, and found them to be as indicated in (6.2).

We emphasize that these calculations do indeed yield an asymptotic expansion, for the error term made in approximating q_0 by the first n terms is easily seen to be $O(1/a^{n+1})$ in each case. □

Theorem 6.3 (p. 48). *Let q_0 be as given in Theorem 6.2. Then, as a tends to ∞,*

$$
q_0 = f(a) + O(1/a^8),
$$

where

(6.7)
$$
\begin{aligned}
f(a) &:= \frac{2}{a - 1 + \sqrt{(a+1)(a+5)}} \\
&= \frac{1}{a} - \frac{1}{a^2} + \frac{2}{a^3} - \frac{6}{a^4} + \frac{21}{a^5} - \frac{79}{a^6} + \frac{311}{a^7} \\
&\quad - \frac{1265}{a^8} + \frac{5275}{a^9} - \frac{22431}{a^{10}} + \frac{96900}{a^{11}} - \cdots.
\end{aligned}
$$

Proof. Expanding $f(a)$ via *Mathematica*, we deduce the Taylor series in a^{-1} given in (6.7). Comparing (6.7) with (6.2), we find that the coefficients of $a^{-j}, 1 \leq j \leq 7$, agree, while the coefficients of a^{-8} differ only by 1. Thus, Ramanujan's first claim in Theorem 6.3 is justified. □

Theorem 6.4 (p. 48). *Let q_0 be as given in Theorem 6.2. Then, as a tends to ∞,*

$$
q_0 = g(a) + O(1/a^{11}),
$$

where

(6.8)
$$
\begin{aligned}
g(a) &:= \frac{1}{\dfrac{a - 1 + \sqrt{(a+1)(a+5)}}{2} + \left(\dfrac{a + 3 - \sqrt{(a+1)(a+5)}}{a - 1 + \sqrt{(a+1)(a+5)}} \right)^3} \\
&= \frac{1}{a} - \frac{1}{a^2} + \frac{2}{a^3} - \frac{6}{a^4} + \frac{21}{a^5} - \frac{79}{a^6} + \frac{311}{a^7} \\
&\quad - \frac{1266}{a^8} + \frac{5289}{a^9} - \frac{22553}{a^{10}} + \frac{97760}{a^{11}} - \cdots.
\end{aligned}
$$

Proof. Expanding $g(a)$ in a Taylor series in a^{-1} with the help of *Mathematica*, we establish the expansion in (6.8). Comparing (6.8) with (6.2), we find that the coefficients of the $a^{-j}, 1 \le j \le 10$, agree, while the coefficients of a^{-11} differ only by 3. Thus, the first assertion in Theorem 6.4 follows. □

In fact, in both the expansions (6.2) and (6.7), Ramanujan calculated just the first ten terms. Our statement of Theorem 6.4 is stronger than that recorded by Ramanujan, who merely claimed that (in different notation) "$q_0 = g(a)$." Undoubtedly, however, he calculated the expansion (6.8). We calculated eleven terms in each expansion for the purpose of comparing accuracies.

7. Explicit Formulas Arising from Modular Equations

Theorem 7.1 (p. 205). *Let* $\omega = \exp(2\pi i/3)$, $u = R(q)$, *and* $v = R(q^2)$. *If*

$$R := \frac{f^3(-q)}{\sqrt{q}f^3(-q^5)} = \sqrt{\frac{1}{u^5} - 11 - u^5},$$

then

$$(7.1) \quad -3v = u^2 + \omega \left(u^6 + 18u + 3iu\sqrt{3}R \right)^{1/3} + \omega^2 \left(u^6 + 18u - 3iu\sqrt{3}R \right)^{1/3}.$$

If

$$R := \frac{f^3(-q^2)}{qf^3(-q^{10})} = \sqrt{\frac{1}{v^5} - 11 - v^5},$$

then

$$(7.2) \quad -3u = \frac{1}{v^2} + \omega \left(\frac{1}{v^6} - \frac{18}{v} + \frac{3\sqrt{3}\, i}{v}R \right)^{1/3} + \omega^2 \left(\frac{1}{v^6} - \frac{18}{v} - \frac{3\sqrt{3}i}{v}R \right)^{1/3}.$$

Theorem 7.2 (p. 205). *Let* $\omega = \exp(2\pi i/3)$, $u = R(q)$, *and* $v = R(q^3)$. *If*

$$R := \frac{f^2(-q^3)}{qf^2(-q^{15})} = \left(\frac{1}{v^5} - 11 - v^5 \right)^{1/3},$$

then

$$(7.3) \quad 4u = -\frac{1}{v^3} - \sqrt{\frac{1}{v^6} - \frac{8+4R}{v}} + \sqrt{\frac{1}{v^6} - \frac{8+4R\omega}{v}} + \sqrt{\frac{1}{v^6} - \frac{8+4R\omega^2}{v}}.$$

If

$$R := \frac{f^2(-q)}{q^{1/3}f^2(-q^5)} = \left(\frac{1}{u^5} - 11 - u^5 \right)^{1/3},$$

then

$$(7.4) \quad 4v = u^3 - \sqrt{u^6 + u(8+4R)} + \sqrt{u^6 + u(8+4R\omega)} + \sqrt{u^6 + u(8+4R\omega^2)}.$$

The four different definitions of R given in Theorems 7.1 and 7.2 arise from (1.7).

To prove the two formulas (7.1) and (7.2), we employ (2.5). Note that this polynomial is cubic in each of u and v. To prove (7.1), we use Cardan's method [31, pp. 84–86] to solve for u in terms of v, and we similarly employ Cardan's method to prove (7.2). However, the calculations are nontrivial, and care must be taken to determine which of the three roots is the correct one in each case. To do this, we require a careful examination of the three roots near $q = 0$.

To prove (7.3) and (7.4), we use Ramanujan's modular equation relating $u = R(q)$ and $v = R(q^3)$, namely,

$$(7.5) \qquad\qquad (v - u^3)(1 + uv^3) = 3u^2 v^2,$$

which is found on page 321 in Ramanujan's second notebook [24] and on page 365 in the publication of his lost notebook [26]. See also [4, p. 27, Entry 20] and [6, Chap. 32, Entry 3]. The only proof in the literature is due to Rogers [28]. Observe that (7.5) is quartic in each of u and v. We thus use Ferrari's method [31, pp. 94–96] to solve for each of u and v. As above, an examination of the roots in a neighborhood of $q = 0$ guides us to the correct root in each case.

Complete proofs of Theorems 7.1 and 7.2 will be given in [3].

8. A Finite Generalized Rogers–Ramanujan Continued Fraction

Theorem 8.1 (p. 54). *For each positive integer n,*

$$(8.1) \qquad \begin{aligned} &1 + \frac{aq}{1} + \frac{a^2 q^4}{1} + \frac{a^2 q^8}{1} + \frac{a^2 q^{12}}{1} + \cdots + \frac{a^2 q^{4(n-1)}}{1} \\ &= \frac{1}{1} - \frac{aq}{1} + \frac{aq}{1} - \frac{aq^3}{1} + \frac{aq^3}{1} - \cdots - \frac{aq^{2n-1}}{1} + \frac{aq^{2n-1}}{1}, \end{aligned}$$

where for $n = 1$ the left side of (8.1) is understood to equal $1 + aq$.

Proof. We use induction on n. For $n = 1$, both sides of (8.1) are equal to $1 + aq$, and for $n = 2$ both sides of (8.1) are equal to

$$\frac{1 + aq + a^2 q^4}{1 + a^2 q^4}.$$

Now assume that (8.1) is valid with n replaced by $n - 1$, and in this inductive assumption replace a by aq^2. Thus,

$$(8.2) \qquad \begin{aligned} &1 + \frac{aq^3}{1} + \frac{a^2 q^8}{1} + \frac{a^2 q^{12}}{1} + \cdots + \frac{a^2 q^{4(n-1)}}{1} \\ &= \frac{1}{1} - \frac{aq^3}{1} + \frac{aq^3}{1} - \frac{aq^5}{1} + \frac{aq^5}{1} - \cdots - \frac{aq^{2n-1}}{1} + \frac{aq^{2n-1}}{1}. \end{aligned}$$

Let

$$(8.3) \qquad\qquad S := 1 + \frac{a^2 q^8}{1} + \frac{a^2 q^{12}}{1} + \cdots + \frac{a^2 q^{4(n-1)}}{1}.$$

Multiplying both sides of (8.2) by aq, we see that

$$(8.4) \qquad aq\left(1 + \frac{aq^3}{S}\right) = \frac{aq}{1} - \frac{aq^3}{1} + \frac{aq^3}{1} - \cdots - \frac{aq^{2n-1}}{1} + \frac{aq^{2n-1}}{1}.$$

Therefore, by (8.4),

$$\frac{1}{1-}\frac{aq}{1+}\frac{aq}{1-}\frac{aq^3}{1+}\frac{aq^3}{1-}\cdots-\frac{aq^{2n-1}}{1}+\frac{aq^{2n-1}}{1}$$

$$= \frac{1}{1-}\frac{aq}{1+aq(1+aq^3/S)}$$

$$= \frac{S+a^2q^4+aqS}{S+a^2q^4}$$

$$= 1+\frac{aqS}{S+a^2q^4}$$

$$= 1+\frac{aq}{1+a^2q^4/S}$$

$$= 1+\frac{aq}{1+}\frac{a^2q^4}{1}+\frac{a^2q^8}{1}+\cdots+\frac{a^2q^{4(n-1)}}{1},$$

where we employed (8.3) in the last step. This completes the proof.   □

9. Finite Rogers–Ramanujan Continued Fractions and Class Invariants

At the bottom of page 47, Ramanujan claims that particular zeros of certain finite Rogers–Ramanujan continued fractions, or similar continued fractions, involve class invariants or singular moduli. For detailed accounts of Ramanujan's work on class invariants and singular moduli, see two papers by Berndt, Chan, and Zhang [9], [10] and Berndt's book [6, Chap. 34]. We present here only the basic definitions and facts that are needed to describe and prove Ramanujan's results in this section.
Let

$$(9.1) \qquad\qquad \chi(q) := (-q;q^2)_\infty, \qquad |q| < 1.$$

If $q = q_n := \exp(-\pi\sqrt{n})$, for some positive rational number n, then the class invariant G_n is defined by

$$(9.2) \qquad\qquad G_n := 2^{-1/4}q_n^{-1/24}\chi(q_n).$$

Let $k := k(q), 0 < k < 1$, denote the modulus, and let $k' = \sqrt{1-k^2}$ denote the complementary modulus. In particular, if $q = q_n$, then $k(q_n) =: k_n$ is called a singular modulus. Also put $k'_n := \sqrt{1-k_n^2}$. Let $K = K(q)$ and $K' = K(q')$ denote complete elliptic integrals of the first kind. If $q = \exp(-\pi K'/K)$, then $\chi(q) = 2^{-1/6}(kk'/q^2)^{-1/12}$ [5, p. 124]. In particular, if $K'/K = \sqrt{n}$, then

$$(9.3) \qquad\qquad G_n = (2k_nk'_n)^{-1/12}.$$

Theorem 9.1 (p. 47). *If $K'/K = \sqrt{47}$, and $t := t_{47} := 2^{1/3}(k_{47}k'_{47})^{1/12}$, then*

$$(9.4) \qquad\qquad 1-\frac{t}{1-}\frac{t^2}{1-}\frac{t^3}{1-}\frac{t^4}{1} = 0.$$

Furthermore,

$$(9.5) \qquad\qquad t_{47} = \sqrt{2}e^{-\pi\sqrt{47}/24}(q_{47};-q_{47})_\infty.$$

Proof. First, from (9.3), it is easy to see that $G_{47} = 2^{1/4}t_{47}^{-1}$. Using (9.1), (9.2), and Euler's identity

$$(9.6) \qquad \frac{1}{(-q;q^2)_\infty} = (q;-q)_\infty,$$

we readily deduce (9.5).

Now from either Weber's treatise [34, p. 723] or Ramanujan's first notebook [24, p. 234], if

$$(9.7) \qquad \sqrt{2}x = e^{\pi\sqrt{47}/24}(-q_{47};q_{47}^2)_\infty,$$

then

$$x^5 = (1+x)(1+x+x^2).$$

Hence, from (9.5)–(9.7), $t = 1/x$. Thus, t satisfies the equation

$$\left(\frac{1}{t}\right)^5 = \left(1+\frac{1}{t}\right)\left(1+\frac{1}{t}+\frac{1}{t^2}\right),$$

i.e.,

$$(9.8) \qquad t^5 + 2t^4 + 2t^3 + t^2 - 1 = 0.$$

Multiply both sides of (9.8) by $(t-1)$ to deduce that

$$(9.9) \qquad t^6 + t^5 - t^3 - t^2 - t + 1 = 0.$$

However, a brief calculation shows that (9.9) is equivalent to (9.4), and this completes the proof. □

Theorem 9.2 (p. 47). *Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', ℓ, and ℓ', respectively. If $K'/K = \sqrt{39}, L'/L = \sqrt{13/3}$, and $t := t_{39} := (k_{39}k'_{39}/\ell_{13/3}\ell'_{13/3})^{1/12}$, then*

$$(9.10) \qquad 1 - \frac{t}{1-}\frac{t^2}{1-}\frac{t^3}{1} = 0.$$

Moreover,

$$(9.11) \qquad t_{39} = e^{-\pi\sqrt{13/3}/12}\frac{(-q_{13/3};q_{13/3}^2)_\infty}{(-q_{13/3}^3;q_{13/3}^6)_\infty}.$$

Ramanujan, observing that each factor in the denominator of (9.11) is cancelled by a corresponding factor in the numerator, wrote (9.11) as a single infinite product.

Proof. By (9.3) and (9.2),

$$(9.12) \qquad t_{39} = \frac{G_{13/3}}{G_{39}} = \frac{q_{13/3}^{-1/24}\chi(q_{13/3})}{q_{39}^{-1/24}\chi(q_{39})},$$

from which, by (9.1), (9.11) trivially follows.

From either Weber's text [34, p. 722] or Ramanujan's notebooks [24, vol. 1, p. 305, or vol. 2, p. 295],

$$(9.13) \qquad G_{39} = 2^{1/4}\left(\frac{\sqrt{13}+3}{2}\right)^{1/6}\left(\sqrt{\frac{5+\sqrt{13}}{8}}+\sqrt{\frac{\sqrt{13}-3}{8}}\right).$$

The class invariant $G_{13/3}$ can be determined from (9.13) and a certain modular equation of degree 3 [9, Lemma 3.3]. Accordingly, we find that

$$(9.14) \qquad G_{13/3} = 2^{1/4} \left(\frac{\sqrt{13}+3}{2} \right)^{1/6} \left(\sqrt{\frac{5+\sqrt{13}}{8}} - \sqrt{\frac{\sqrt{13}-3}{8}} \right).$$

Thus, from (9.12)–(9.14),

$$t_{39} = \left(\sqrt{\frac{5+\sqrt{13}}{8}} - \sqrt{\frac{\sqrt{13}-3}{8}} \right)^2.$$

It is now easily checked that t_{39} is a root of the polynomial equation

$$(9.15) \qquad\qquad t^4 - t^3 - t^2 - t + 1 = 0.$$

Observing that (9.10) and (9.15) are equivalent, we complete the proof. $\qquad\square$

Theorem 9.3 (p. 47). *If $t := t_{23} := 2^{1/3}(k_{23}k'_{23})^{1/12}$, then*

$$(9.16) \qquad\qquad 1 - \frac{t^2}{1} \frac{t^3}{-1} = 0.$$

The value of t in this result was, in fact, not given by Ramanujan. If $F(t)$ denotes the continued fraction in (9.16), then $F(t)$ is not a finite Rogers–Ramanujan continued fraction. However, $1 - t/F(t)$ is a finite Rogers–Ramanujan continued fraction.

Proof. As we argued in the proof of Theorem 9.1, $G_{23} = 2^{1/4}t_{23}^{-1}$. From Weber's tables [34, p. 722] or Ramanujan's notebooks [24, vol. 1, pp. 295, 345, 351; vol. 2, p. 294], if $G_{23} = 2^{1/4}x$, then

$$x^3 - x - 1 = 0.$$

Thus, $t = t_{23} = 1/x$ and

$$(9.17) \qquad\qquad t^3 + t^2 - 1 = 0.$$

It is easy to see that (9.17) and (9.16) are equivalent, and so this completes the proof. $\qquad\square$

Theorem 9.4 (p. 47). *If $t := t_{31} := 2^{1/3}(k_{31}k'_{31})^{1/12}$, then*

$$(9.18) \qquad\qquad 1 - \frac{t}{1} \frac{t^3}{-1} = 0.$$

As with Theorem 9.3, Ramanujan did not provide the definition of t in Theorem 9.4. Also, the continued fraction in (9.18) is not a finite Rogers–Ramanujan continued fraction.

Proof. By a now familiar argument, $G_{31} = 2^{1/4}t_{31}^{-1}$. From Weber's tables [34, p. 722] or Ramanujan's notebooks [24, vol. 1, pp. 296, 345, 351; vol. 2, p. 295], if $G_{31} = 2^{1/4}x$, then

$$x^3 - x^2 - 1 = 0.$$

Thus, $t_{31} = 1/x$ and

$$(9.19) \qquad\qquad t^3 + t - 1 = 0.$$

Clearly, (9.19) and (9.18) are equivalent, and so the proof is complete. $\qquad\square$

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